The Logic of Confirmation and Theory Assessment

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1. Hempel's conditions of adequacy

In his "Studies in the Logic of Confirmation" (1945) Carl G. Hempel presented the following conditions of adequacy for any relation of confirmation $\mid\!\sim\ \subseteq \mathcal{L}\times\mathcal{L}$ on some language (set of well formed formulas) $\mathcal{L}$ (names for 3.1 and 3.2 added). For any (observation report) $E \in \mathcal{L}$ and any (hypothesis or theory) $H \in \mathcal{L}$:

1. Entailment Condition: $E \vdash H \;\Rightarrow\; E \mid\!\sim H$
2. Consequence Condition: $\{H: E \mid\!\sim H\} \vdash H' \;\Rightarrow\; E \mid\!\sim H'$
    - 2.1 Special Consequence Cond.: $E \mid\!\sim H,\; H \vdash H' \;\Rightarrow\; E \mid\!\sim H'$
    - 2.2 Equivalence Condition: $E \mid\!\sim H,\; H \dashv\vdash H' \;\Rightarrow\; E \mid\!\sim H'$
3. Consistency Condition: $\{E\}\cup\{H: E \mid\!\sim H\} \nvdash \bot$
    - 3.1 Special C.C.: $E \nvdash \bot,\; E \mid\!\sim H,\; H \vdash \neg H' \;\Rightarrow\; E \mid\!\not\sim H'$
    - 3.2 Consistent Selectivity: $E \nvdash \bot,\; E \mid\!\sim H \;\Rightarrow\; E \mid\!\not\sim \neg H$
4. Converse Consequence Cond.: $E \mid\!\sim H,\; H' \vdash H \;\Rightarrow\; E \mid\!\sim H'$

($\vdash\ \subseteq \wp(\mathcal{L})\times\mathcal{L}$ is the classical deducibility relation, and singletons of wffs are identified with the wff they contain.) 2 entails 2.1 which in turn entails 2.2; similarly for 3. Hempel (1945, p. 104) then showed that the conjunction of 1, 2, and 4 entails his triviality result that every sentence (observation report) E confirms every sentence (hypothesis or theory) H, i.e. for all $E, H \in \mathcal{L}$: $E \mid\!\sim H$. This is clear since the conjunction of 1 and 4 already implies this: By the Entailment Condition, E confirms $E\vee H$; as $H \vdash E\vee H$, the Converse Consequence Condition yields that E confirms H.

Since Hempel's negative result, there has hardly been any progress in developing a logic of confirmation.[1] One reason for this seems to be that up to now

[1] The exceptions I know of are (Flach, 2000; Milne, 2000; Zwirn & Zwirn, 1996). Roughly, Zwirn & Zwirn (1996) argue that there is no unified logic of confirmation (taking into account all of the partly conflicting aspects of confirmation); Flach (2000) argues that there are two logics of "induction", as he calls it, viz. confirmatory and explicatory induction (corresponding to Hempel's conditions 1–3 and 4, respectively); and Milne (2000) argues that there is a logic of confirmation (namely the logic of positive relevance), but that it does not deserve to be called a logic.

the predominant view on Hempel's conditions is the analysis Carnap gave in his *Logical Foundations of Probability* (1962, §87).

2. Carnap's analyis of Hempel's conditions

In analyzing the Consequence Condition, Carnap argues that

> Hempel has in mind as explicandum the following relation: 'the degree of confirmation of H by E is greater than r', where r is a fixed value, perhaps 0 or 1/2.
> (Carnap, 1962, p. 475; notation adapted)

In discussing the Consistency Condition, Carnap mentions that

> Hempel himself shows that a set of physical measurements may confirm several quantitative hypotheses which are incompatible with each other (p. 106). This seems to me a clear refutation of [3.1]. ... What may be the reasons that have led Hempel to the consistency conditions [3.1] and [3]? He regards it as a great advantage of any explicatum satisfying [3] "that is sets a limit, so to speak, to the strength of the hypotheses which can be confirmed by given evidence" ... This argument does not seem to have any plausibility for *our* explicandum,
> (Carnap, 1962, pp. 476–7; emphasis in original)

which is the concept of positive probabilistic relevance, or "initially confirming evidence" as Carnap says in §86 of his (1962);

> [b]ut it is plausible for the second explicandum mentioned earlier: the degree of confirmation exceeding a fixed value r. Therefore we may perhaps assume that Hempel's acceptance of the consistency condition is due again to an inadvertant shift to the second explicandum.
> (Carnap, 1962, pp. 477–8)

Carnap's analysis can be summarized as follows: In presenting his first three conditions of adequacy Hempel was mixing up two distinct concepts of confirmation, two distinct explicanda in Carnap's terminology, viz.

1. the concept of incremental confirmation (positive probabilistic relevance, initially confirming evidence) according to which E confirms H iff E (has non-zero probability and) increases the probability of H, $\Pr(H \mid E) > \Pr(H)$, and

2. the concept of absolute confirmation according to which E confirms H iff the probability of H given E is greater than some value r, $\Pr(H \mid E) > r$.

The special versions of Hempel's second and third condition, 2.1 and 3.1, respectively, hold true for the second explicandum (for $r \geq 1/2$), but they do not hold true for the first explicandum. On the other hand, Hempel's first condition holds true for the first explicandum, but it does so only in a *qualified* form (cf. Carnap, 1962, p. 473) – namely only if E is not assigned probability 0, and H is not already assigned probability 1.

This, however, means that Hempel first had in mind the explicandum of incremental confirmation for the Entailment Condition; then he had in mind the explicandum of absolute confirmation for the Consequence and the Consistency Conditions 2.1 and 3.1, respectively; and then, when Hempel presented the Converse Consequence Condition, he got completely confused, and had in mind still another explicandum or concept of confirmation.[2] Apart from not being very charitable, Carnap's reading of Hempel also leaves open the question what the third explicandum might have been.

3. Conflicting concepts of confirmation

The following two notions are central to the plausibility-informativeness theory of theory assessment first presented in Huber (2004) and further developed in Huber (in preparation):

Definition 1 *A relation* $\mid\!\sim \subseteq \mathcal{L}\times\mathcal{L}$ *on a language* $\mathcal{L}$ *is an informativeness relation iff for all* $E, H, H' \in \mathcal{L}$:

$$E \mid\!\sim H, \quad H' \vdash H \quad \Rightarrow \quad E \mid\!\sim H'.$$

$\mid\!\sim \subseteq \mathcal{L}\times\mathcal{L}$ *is a plausibility relation on* $\mathcal{L}$ *iff for all* $E, H, H' \in \mathcal{L}$:

$$E \mid\!\sim H, \quad H \vdash H' \quad \Rightarrow \quad E \mid\!\sim H'.$$

The idea is that a sentence or proposition is the more informative, the more possibilities it excludes. Hence, the logically stronger a sentence, the more informative it is. On the other hand, a sentence is more plausible the fewer possibilities it excludes. Hence, the logically weaker a sentence, the more plausible it is. The qualitative counterparts of these two principles are the defining clauses above:

[2] Neither the first nor the second explicandum satisfies the Converse Consequence Condition.

If H is informative relative to E, then so is any logically stronger H'. Similarly, if E is plausible relative to E, then so is any logically weaker H'.

The two main approaches to confirmation that have been put forth in the last century are qualitative *Hypothetico-Deductivism* HD and quantitative probabilistic *Inductive Logic* IL. According to HD, E HD-confirms H iff H logically implies E (in some suitable way that depends on the version of HD under consideration). According to IL, the degree of *absolute* confirmation of H by E equals the (logical) probability of H given E, $\Pr(H \mid E)$. The natural *qualitative* counterpart of this quantitative notion is that *E absolutely IL-confirms H* iff $\Pr(H \mid E) > r$, for some $r \in [.5, 1)$ (this is Carnap's second explicandum).

As noted above, this is not the way Carnap defined qualitative IL-confirmation in chapter VII of his (1962). There he required that E raises the probability of H, $\Pr(H \mid E) > \Pr(H)$, in order for E to qualitatively IL-confirm H. Nevertheless, the above seems to be the natural qualitative counterpart of the degree of absolute confirmation. The reason is that later on, the difference between $\Pr(H \mid E)$ and $\Pr(H)$ – however it is measured (cf. Fitelson, 2001) – was taken as the degree of *incremental confirmation*, and Carnap's proposal is the natural qualitative counterpart of this notion of incremental confirmation. In order to separate these two notions, let us say that E incrementally confirms H iff $\Pr(H \mid E) > \Pr(H)$.

HD and IL are based on two *conflicting* concepts of confirmation. HD-confirmation *increases*, whereas absolute IL-confirmation *decreases* with the logical strength of the theory to be assessed. More precisely, if E HD-confirms H and $H' \vdash H$, then E HD-confirms H'. So, as a matter of fact, HD-confirmation aims at logically strong theories – HD-confirmation is an informativeness relation. On the other hand, if E absolutely IL-confirms H (to some degree r) and $H \vdash H'$, then E absolutely IL-confirms H' (to at least the same degree $s \geq r$). So absolute IL-confirmation aims at logically weak theories – absolute IL-confirmation is a plausibility relation.

The epistemic virtues behind these two notions are *informativeness* on the one hand and *truth* on the other hand. First, we want to know what is going on "out there", and hence we aim at true theories – more precisely, at theories that are true in the world we are in. Second, we want to know as much as possible about what is going on out there, and so we aim at informative theories – more precisely, at theories that inform us about the world we are in. But usually we do not know which world we are in. All we have are some data or truth conditions, so we base our evaluation of the theory we are concerned with on the plausibility that the theory is true in the actual world given that the actual world makes the data true and on how much the theory informs us about the actual world given that the actual world makes the data true.

If one of two theories logically implies the other, the logically stronger theory excludes all the possibilities excluded by the logically weaker theory. The logically stronger theory is thus at least as informative as the logically weaker one. On the other hand, the logically weaker theory is at least as plausible as the logically stronger theory, because all possibilities making the logically stronger theory true also make the logically weaker theory true. This is the sense in which the two concepts underlying HD and IL, respectively, are *conflicting*.

4. Hempel vindicated

Turning back to Hempel's conditions, note first that Carnap's second explicandum satisfies the Entailment Condition *without* the second qualification: If $E \vdash H$, then $\Pr(H \mid E) = 1 > r$, for any value $r < 1$, provided E does not have probability 0.

So the following more charitable reading of Hempel seems plausible: When presenting his first three conditions, Hempel had in mind Carnap's second explicandum, the concept of absolute confirmation, or more generally, a plausibility relation. But then, when discussing the Converse Consequence Condition, Hempel also felt the need for a second concept of confirmation aiming at informative theories.

Given that it was the Converse Consequence Condition which Hempel gave up in his "Studies", the present analysis makes perfect sense of his argumentation: Though he felt the need for two concepts of confirmation, Hempel also realized that these two concepts are *conflicting* (this is the content of his triviality result), and so he abandoned informativeness in favour of plausibility.

5. The logic of theory assessment

However, in a sense, one can have Hempel's cake and eat it too: There is a logic of confirmation – or rather, theory assessment – that takes into account both of these two conflicting concepts. Roughly speaking, HD says that a good theory is informative, whereas IL says that a good theory is plausible or true. The driving force behind Hempel's conditions is the insight that *a good theory is both true and informative*. Hence, in assessing a given theory by the available data, one should account for these two conflicting aspects.

According to this logic, a sentence or proposition H is *an acceptable theory* or *in the pool of reasonable theories* for evidence E iff H is at least as plausible as and more informative than its negation relative to E, or H is more plausible than and at least as informative as its negation relative to E. This is spelt out more precisely in terms ranking functions (Spohn, 1988, 1990).

5.1 Assessment models

Let us first fix some terminology. A *language* $\mathcal{L}$ is a countable set of (propositional or first order) wffs that is closed under the propositional connectives $\neg$ and $\wedge$ ($\vee$, $\rightarrow$, $\leftrightarrow$ are defined as usual). A language is not required to be closed under the quantifiers. $Mod_{\mathcal{L}} = Mod$ is the set of all models for $\mathcal{L}$. $\models \subseteq Mod \times \mathcal{L}$ is the classical satisfaction relation, and, for $\alpha \in \mathcal{L}$, $Mod(\alpha) = \{\omega \in Mod: \omega \models \alpha\}$. $\models$ is compact – a set of wffs is satisfiable iff all its finite subsets are – and such that $\omega \models \alpha$ iff $\omega \not\models \neg\alpha$ and $Mod(\alpha \wedge \beta) = Mod(\alpha) \cap Mod(\beta)$.

Let W be a non-empty set of possibilities, and let $\mathcal{A}$ be a field over W, i.e. a set of subsets of W containing the empty set and closed under complementation and finite intersections. A function κ from W into the set of natural numbers N extended by ∞ is a *pointwise ranking function* iff at least one $\omega \in W$ is assigned κ-rank 0, i.e. $\kappa^{-1}(0) \neq \emptyset$. $\kappa: \mathcal{A} \rightarrow N \cup \{\infty\}$ is a *general ranking function* iff for all $A, B \in \mathcal{A}$:

1. $\kappa(W) = 0$
2. $\kappa(\emptyset) = \infty$
3. $\kappa(A \cup B) = \min\{\kappa(A), \kappa(B)\}$

The conditional rank of B given A, $\kappa(B|A)$, is defined as

$$4.\ \ \kappa(B \mid A) = \begin{cases} \kappa(A \cap B) - \kappa(A), & \text{if} \quad \kappa(A) < \infty, \\ 0, & \text{if} \quad \kappa(A) = \infty. \end{cases}$$

(Goldszmidt & Pearl, 1996, p. 63, stipulate $\kappa(B \mid A) = \infty$ for $\kappa(A) = \infty$.) κ is *regular* iff $\kappa(A) < \kappa(\emptyset)$ for each non-empty $A \in \mathcal{A}$. A pointwise ranking function $\kappa: W \rightarrow N \cup \{\infty\}$ is uniquely extended to a general ranking function $\kappa_{\min}$: $\mathcal{A} \rightarrow N \cup \{\infty\}$ by defining, for each $A \in \mathcal{A}$:

$$\kappa_{\min}(A) = \begin{cases} \min\{\kappa(\omega) : \omega \in A\}, & \text{if} \quad A \neq \emptyset, \\ \infty, & \text{if} \quad A = \emptyset. \end{cases}$$

"Ranks represent degrees" – or rather, grades – "of disbelief" (Spohn, 1999, p. 6). Whereas a high probability indicates a high degree of belief, a high rank indicates a high grade of disbelief.

A ranking space $\langle W, \mathcal{A}, \kappa\rangle$ is a (*rank-theoretic*) *assessment model* for the language $\mathcal{L}$ iff $W = Mod_{\mathcal{L}}$, $Mod(\alpha) \in \mathcal{A}$ for each $\alpha \in \mathcal{L}$, and κ is a regular general ranking function on $\mathcal{A}$. In general there are many different fields $\mathcal{A}$ for a particular set W. So there are many different assessment models for a particular language $\mathcal{L}$, but the set W is the same for all of them. The "consequence relation" $\vdash_{\kappa} \subseteq \mathcal{L} \times \mathcal{L}$ defined by an assessment model $\langle W, \mathcal{A}, \kappa\rangle$ for $\mathcal{L}$ is given as follows:

$$
\begin{aligned}
\alpha \mathrel{|\!\!\sim}_{\kappa} \beta \;\Leftrightarrow\; & [\kappa(Mod(\beta) \mid Mod(\alpha)) < \kappa(\overline{\mathrm{Mod}(\beta)} \mid Mod(\alpha)) \;\;\& \\
& \kappa(\overline{\mathrm{Mod}(\beta)} \mid \overline{Mod(\alpha)}) \leq \kappa(Mod(\beta) \mid \overline{Mod(\alpha)})] \\
& \text{or} \\
& [\kappa(Mod(\beta) \mid Mod(\alpha) \leq \kappa(\overline{Mod(\beta)} \mid Mod(\alpha)) \;\;\& \\
& \kappa(\overline{Mod(\beta)} \mid \overline{Mod(\alpha)}) < \kappa(Mod(\beta) \mid \overline{Mod(\alpha)})] \\
\Leftrightarrow\; & [\kappa(Mod(\alpha\wedge\beta)) < \kappa(Mod(\neg\beta\wedge\alpha)) \;\;\& \\
& \kappa(Mod(\neg\beta\wedge\neg\alpha)) \leq \kappa(Mod(\beta\wedge\neg\alpha))] \\
& \text{or} \\
& [\kappa(Mod(\beta\wedge\alpha)) \leq \kappa(Mod(\neg\beta\wedge\alpha)) \;\;\& \\
& \kappa(Mod(\neg\beta\wedge\neg\alpha)) < \kappa(Mod(\beta\wedge\neg\alpha))].
\end{aligned}
$$

This reads as follows: β is an acceptable theory for α (in the sense of $\langle W, \mathcal{A}, \kappa\rangle$) iff β is at least as plausible (in the sense of $\langle W, \mathcal{A}, \kappa\rangle$) given α as is ¬β given α, and β informs us more about α (in the sense of $\langle W, \mathcal{A}, \kappa\rangle$) than does ¬β; or β is more plausible given α than is ¬β given α, and β informs us at least as much about α as does ¬β.

In the following we employ the Gabbay-Makinson-KLM framework (Gabbay, 1985; Makinson, 1989; Kraus, Lehmann, & Magidor, 1990) and present a list of properties such that the consequence relation $\mathrel{|\!\!\sim}_{\kappa}$ defined by any assessment model for any language $\mathcal{L}$ satisfies these properties (soundness). Then we show that the converse is also true: For each relation $\mathrel{|\!\!\sim} \subseteq \mathcal{L}\times\mathcal{L}$ on some language $\mathcal{L}$ satisfying these properties there is an assessment model $\langle W, \mathcal{A}, \kappa\rangle$ for $\mathcal{L}$ such that $\mathrel{|\!\!\sim} = \mathrel{|\!\!\sim}_{\kappa}$ (completeness).

5.2 Assessment relations

A relation $\mathrel{|\!\!\sim} \subseteq \mathcal{L}\times\mathcal{L}$ is a (*rank-theoretic*) *assessment relation* on the language $\mathcal{L}$ iff $\mathrel{|\!\!\sim}$ satisfies the following principles:

1.	$\alpha \mathrel{|\!\!\sim} \alpha$	Reflexivity*
2.	$\alpha \mathrel{|\!\!\sim} \beta,\;\; \alpha \dashv\vdash \gamma \;\Rightarrow\; \gamma \mathrel{|\!\!\sim} \beta$	Left Logical Equivalence*
3.	$\alpha \mathrel{|\!\!\sim} \beta,\;\; \beta \dashv\vdash \gamma \;\Rightarrow\; \alpha \mathrel{|\!\!\sim} \gamma$	Right Logical Equivalence*
4.	$\alpha \mathrel{|\!\!\sim} \beta \;\Rightarrow\; \alpha \mathrel{|\!\!\sim} \alpha\wedge\beta$	Weak Composition*
5.	$\alpha \mathrel{|\!\!\sim} \beta \;\Rightarrow\; \neg\alpha \mathrel{|\!\!\sim} \neg\beta$	Loveliness and Likeliness
6.	$\nvdash \alpha\vee\beta \;\Rightarrow\; \alpha\vee\beta \mathrel{|\!\!\sim} \alpha$ or $\alpha\vee\beta \mathrel{|\!\!\sim} \beta$	Either-Or
7.	$\alpha\vee\beta \mathrel{|\!\!\not\sim} \alpha,\;\; \nvdash \alpha\vee\beta \;\Rightarrow\; \alpha\vee\neg\alpha \mathrel{|\!\!\sim} \neg\alpha$	Negation
8.	$\alpha\wedge\neg\alpha \mathrel{|\!\!\sim} \alpha,\;\; \alpha\vee\beta \mathrel{|\!\!\sim} \alpha \;\Rightarrow\; \alpha\wedge\neg\alpha \mathrel{|\!\!\sim} \beta$	Down
9.	$\alpha \mathrel{|\!\!\sim} \alpha\wedge\beta,\;\; \alpha \mathrel{|\!\!\sim} \alpha\vee\beta \;\Rightarrow\; \alpha \mathrel{|\!\!\not\sim} \neg\beta$	TBA I
10.	$\alpha \mathrel{|\!\!\not\sim} \alpha\wedge\neg\beta,\;\; \alpha \mathrel{|\!\!\sim} \alpha\vee\beta,\;\; \nvdash\alpha \;\Rightarrow\; \alpha \mathrel{|\!\!\sim} \beta$	TBA II
11.	$\alpha\vee\beta \mathrel{|\!\!\sim} \alpha,\;\; \beta\vee\gamma \mathrel{|\!\!\sim} \beta,\;\; \nvdash \alpha\vee\gamma \;\Rightarrow\; \alpha\vee\gamma \mathrel{|\!\!\sim} \alpha$	quasi Nr 21
12.	$\alpha\vee\beta \mathrel{|\!\!\sim} \alpha,\;\; \beta\vee\gamma \mathrel{|\!\!\sim} \beta,\;\; \vdash \alpha\vee\gamma \;\Rightarrow\; \alpha\vee\gamma \mathrel{|\!\!\sim} \neg\alpha$	supplementary Nr 21

13. $\alpha_i \vee \alpha_{i+1} \mathrel{|\!\sim} \alpha_{i+1}$, $\not\mathrel{|\!\sim} \alpha_i \vee \alpha_j \;\Rightarrow\; \exists n\, \forall m \geq n$: $\alpha_m \vee \alpha_{m+1} \mathrel{|\!\sim} \alpha_m$
Minimum (no strictly $\prec$-decreasing sequence – see below)

The *-starred principles are among the *core principles* in (Zwirn & Zwirn, 1996). Loveliness and Likeliness is not the same as Negation Symmetry in (Milne, 2000). Quasi Nr 21 without the restriction $\not\mathrel{|\!\sim} \alpha \vee \gamma$ is the derived rule (21) of the system **P** in (Kraus, Lehmann, & Magidor, 1990, cf. their lemma 22). Together with supplementary Nr 21 it expresses the transitivity of the $\leq$-relation between natural numbers. In general, for non-tautological $\alpha \vee \beta$, $\alpha \vee \beta \mathrel{|\!\sim} \alpha$ means that the rank of α is not greater than the rank of β, or equivalently, that the rank of α is not greater than, and hence equal to, the rank of $\alpha \vee \beta$. For tautological $\alpha \vee \beta$, $\alpha \vee \beta \mathrel{|\!\sim} \alpha$ means that the rank of α is strictly smaller than that of its negation $\neg\alpha$, which holds iff $\neg\alpha$ has a rank greater than 0.

Here are some derived rules:

14. $\alpha \mathrel{|\!\sim} \beta \;\Rightarrow\; \alpha \mathrel{|\!\sim} \alpha \vee \beta$ — Weak $\vee$-Composition
15. $\alpha \mathrel{|\!\sim} \beta \;\Rightarrow\; \alpha \not\mathrel{|\!\sim} \neg\beta$ — Selectivity*
16. $\alpha \mathrel{|\!\sim} \beta \;\Rightarrow\; \alpha \vee \beta \mathrel{|\!\sim} \beta$ — TBA III
17. $\alpha \vee \neg\alpha \mathrel{|\!\sim} \alpha$, $\alpha \mathrel{|\!\sim} \beta \;\Rightarrow\; \alpha \vee \neg\alpha \mathrel{|\!\sim} \beta$ — Up

Note that Selectivity allows there to be two logically incompatible theories β_1 and β_2 such that both are acceptable given α (cf. Carnap's discussion of Hempel's consistency condition quoted in section 2).

5.3 A representation result

Theorem 1 *The consequence relation* $\mathrel{|\!\sim}_\kappa$ *induced by any (rank-theoretic) assessment model* $\langle Mod_{\mathcal{L}}, \mathcal{A}, \kappa \rangle$ *for any language* $\mathcal{L}$ *is a (rank-theoretic) assessment relation on* $\mathcal{L}$. *Conversely, for each assessment relation* $\mathrel{|\!\sim}$ *on any language* $\mathcal{L}$ *there is a (rank-theoretic) assessment model* $\langle Mod_{\mathcal{L}}, \mathcal{A}, \kappa \rangle$ *such that* $\mathrel{|\!\sim} = \mathrel{|\!\sim}_\kappa$.

Sketch of Proof:
One starts with the given assessment relation $\mathrel{|\!\sim}$ on the language $\mathcal{L}$ and considers the field $\mathcal{A} = \{Mod(\alpha): \alpha \in \mathcal{L}\}$ on the set of possibilities $W = Mod_{\mathcal{L}}$. Using $\mathrel{|\!\sim}$, one defines a weak order $\preceq_{\mathrel{|\!\sim}} =: \preceq$ on $\mathcal{A}$, where $A \preceq B$ is intended to mean $\kappa(A) \leq \kappa(B)$, for the general ranking function κ to be defined on $\mathcal{A}$. $\preceq$ gives rise to a well order on the set of $\cong$-equivalence classes $\mathcal{A}/\cong = \{[A]: A \in \mathcal{A}\}$, where $A \cong B$ iff $A \preceq B$ and $B \preceq A$, and $[A] = \{B \in \mathcal{A}: A \cong B\}$. This implies that the elements of $\mathcal{A}/\cong$ can be written as a sequence. The rank of $A \in \mathcal{A}$ is defined as the index of its equivalence class $[A] \in \mathcal{A}/\cong$ in this sequence. κ so defined is regular and represents $\preceq$, i.e. $A \preceq B$ iff $\kappa(A) \leq \kappa(B)$, for $A, B \in \mathcal{A}$. Furthermore, $\alpha \mathrel{|\!\sim} \beta$

iff $\kappa(Mod(\beta\wedge\alpha)) \leq \kappa(Mod(\neg\beta\wedge\alpha))$ and $\kappa(Mod(\neg\beta\wedge\alpha)) \leq \kappa(Mod(\beta\wedge\neg\alpha))$, where at least one of these inequalities is strict. The proof is finished by applying the extension theorem for rankings on languages. The latter implies that for κ on $\mathcal{A}$ there exists a unique minimal pointwise ranking function κ^* on $W = Mod_{\mathcal{L}}$ such that $\kappa(A)=\min\{\kappa^*(\omega)\colon \omega \in W\}$ for all non-empty $A \in \mathcal{A}$, i.e. $\kappa = \kappa^*_{min}$. □

6. Comparisions and further (non-)principles

Assessment relations satisfy the following principles.

18.	$\not\mid\!\sim \neg\alpha \;\Rightarrow\; \alpha \not\mid\!\sim \alpha\wedge\neg\alpha$	Consistency*
19.	$\not\mid\!\sim \alpha \;\Rightarrow\; \alpha \not\mid\!\sim \alpha\vee\neg\alpha$	Informativeness
20.	$\alpha \mid\!\sim \alpha\rightarrow\beta \;\Rightarrow\; \alpha \mid\!\sim \beta$	Ampliativity I
21.	$\alpha\vee\neg\alpha \mid\!\sim \alpha \;\Rightarrow\; \alpha\vee\beta \mid\!\sim \alpha$	Negation 2
22.	$\alpha \mid\!\sim \beta,\; \alpha \mid\!\sim \gamma \;\Rightarrow\; \alpha \mid\!\sim \beta\wedge\gamma$ or $\alpha \mid\!\sim \beta\vee\gamma$	quasi-Composition
23.	$\alpha\vee\beta\vee\gamma \mid\!\sim \beta\vee\gamma,\; \not\mid\!\sim \alpha\vee\beta,\; \not\mid\!\sim \alpha\vee\gamma \;\Rightarrow\; \alpha\vee\beta \mid\!\sim \beta$ or $\alpha\vee\gamma \mid\!\sim \gamma$	Ranks

The following three principles from (Zwirn & Zwirn, 1996) are not admissible.

i.	$\alpha \mid\!\sim \alpha\wedge\beta \;\Rightarrow\; \alpha \mid\!\sim \beta$	Weak Consequence
ii.	$\alpha \mid\!\sim \beta \;\Rightarrow\; \alpha \mid\!\sim \alpha\rightarrow\beta$	Ampliativity II

Ampliativity II is a special case of

iii.	$\alpha \mid\!\sim \beta,\; \alpha \vdash \beta\leftrightarrow\gamma \;\Rightarrow\; \alpha \mid\!\sim \gamma$	Levi Principle

The Levi Principle requires, among other things, that refuted theories and verified theories, respectively, are all treated the same. It is clear that this does not hold for assessment, because not all verified theories are as uninformative as tautological theories. Given Carnap's discussion of Hempel's consistency condition (quoted in section 2), it is particularly interesting to observe that

iv.	$\alpha \mid\!\sim \beta,\; \beta \vdash \neg\gamma \;\Rightarrow\; \alpha \not\mid\!\sim \gamma$	Strong Selectivity

is not admissible.

6.1 Assessment relations versus explanatory and confirmatory consequence relations

According to Flach (2000, p. 167ff.), any inductive consequence relation satisfies Left Logical Equivalence, Right Logical Equivalence, Verification, Left Reflexivity, Right Reflexivity, Right Extension, and Falsification. F-Consistency (called Consistency by Flach, 2000, p. 168) is equivalent to Falsification, given Left Logical Equivalence (Flach, 2000, Lemma 1). Hence it is also satisfied by any inductive consequence relation.

24.	$\alpha \mathrel{|\!\sim} \beta,\ \alpha\wedge\beta \vdash \gamma \ \Rightarrow\ \alpha\wedge\gamma \mathrel{|\!\sim} \beta$	Verification
25.	$\alpha \mathrel{|\!\sim} \beta \ \Rightarrow\ \alpha \mathrel{|\!\sim} \alpha$	Left Reflexivity
26.	$\alpha \mathrel{|\!\sim} \beta \ \Rightarrow\ \beta \mathrel{|\!\sim} \beta$	Right Reflexivity
27.	$\alpha \mathrel{|\!\sim} \beta,\ \alpha\wedge\beta \vdash \gamma \ \Rightarrow\ \alpha \mathrel{|\!\sim} \beta\wedge\gamma$	Right Extension
v.	$\alpha \mathrel{|\!\sim} \beta,\ \alpha\wedge\beta \vdash \gamma \ \Rightarrow\ \alpha\wedge\neg\gamma \mathrel{|\!\not\sim} \beta$	Falsification
vi.	$\beta \vdash \neg\alpha \ \Rightarrow\ \alpha \mathrel{|\!\not\sim} \beta$	F-Consistency

These principles are satisfied by assessment relations, if Falsification (F-Consistency) is weakened to quasi-Falsification (quasi-F-Consistency).

28.	$\alpha \mathrel{|\!\sim} \beta,\ \alpha\wedge\beta \vdash \gamma,\ \alpha \not\vdash \gamma \ \Rightarrow\ \alpha\wedge\neg\gamma \mathrel{|\!\not\sim} \beta$	quasi-Falsification
29.	$\beta \vdash \neg\alpha,\ \not\vdash \neg\alpha \ \Rightarrow\ \alpha \mathrel{|\!\not\sim} \beta$	quasi-F-Consistency

Left Reflexivity and Right Reflexivity are unconditionally satisfied in the present system. In (Flach, 2000), the antecedents ensure that α and β, respectively, are consistent.

Among inductive consequence relations, Flach distinguishes between consequence relations for explanatory induction and for confirmatory induction. Explanatory induction $\mathrel{|\!\sim}$ is semantically characterised by defining $\alpha \mathrel{|\!\sim}_W \beta$ iff (i) there is an $\omega \in W$ such that $\omega \models \beta$, and (ii) for all $\omega \in W$: $\omega \models \beta\rightarrow\alpha$, where $W \subseteq Mod_{\mathcal{L}}$ and $Mod_{\mathcal{L}}$ is the set of all models for the propositional language $\mathcal{L}$ ($\models\ \subseteq Mod_{\mathcal{L}}\times\mathcal{L}$ is a compact satisfaction relation).

Explanatory induction thus focuses more or less exclusively (apart from demanding β to be *W*-consistent) on the logical strength of β. It satisfies all principles for inductive consequence relations and is syntactically characterised by Explanatory Reflexivity, Left Consistency, Admissible Right Strengthening, Cautious Monotonicity (called Incrementality by Flach, 2000, p. 172), Predictive Convergence, and Conditionalisation.

30.	$\alpha \mathrel{|\!\sim} \alpha,\ \neg\beta \mathrel{|\!\not\sim} \alpha \ \Rightarrow\ \beta \mathrel{|\!\sim} \beta$	Explanatory Reflexivity
31.	$\alpha \mathrel{|\!\sim} \beta \ \Rightarrow\ \neg\alpha \mathrel{|\!\not\sim} \beta$	Left Consistency
vii.	$\alpha \mathrel{|\!\sim} \beta,\ \gamma \mathrel{|\!\sim} \gamma,\ \gamma \vdash \beta \ \Rightarrow\ \alpha \mathrel{|\!\sim} \gamma$	Admissible Right Strengthening

viii.	$\alpha \mid\sim \gamma,\ \beta \mid\sim \gamma \ \Rightarrow\ \alpha\wedge\beta \mid\sim \gamma$	Cautious Monotonicity
ix.	$\alpha\wedge\gamma \mid\sim \beta,\ \alpha \mid\sim \gamma \ \Rightarrow\ \beta \mid\sim \gamma$	Predictive Convergence
x.	$\alpha \mid\sim \beta\wedge\gamma \ \Rightarrow\ \beta\rightarrow\alpha \mid\sim \gamma$	Conditionalisation

Assessment relations satisfy Explanatory Reflexivity and Left Consistency, but they violate Admissible Right Strengthening, Cautious Monotonicity, Predictive Convergence, and Conditionalisation.

Another class of inductive consequence relations is given by what Flach calls confirmatory induction. These are semantically characterised with the help of *confirmatory structures* $W = \langle S, [\cdot], \|\cdot\|\rangle$, where S is a set of semantic objects, and $[\cdot]$ and $\|\cdot\|$ are functions from the propositional language $\mathcal{L}$ into the powerset of S. $W = \langle S, [\cdot], \|\cdot\|\rangle$ is *simple* just in case for all $\alpha, \beta \in \mathcal{L}$: $[\alpha] \subseteq \|\alpha\|$, $\|\alpha\wedge\beta\| = \|\alpha\|\cap\|\beta\|$, $\|\neg\alpha\| = S\setminus\|\alpha\|$, and $\|\alpha\| = S$ iff $\vDash \alpha$. Given such a confirmatory structure W, the *closed confirmatory consequence relation* $\mid\sim_W$ defined by W simply is the usual KLM consequence relation with the additional requirement that α be consistent (in the sense of $[\cdot]$), i.e. $\alpha \mid\sim_W \beta$ iff $\emptyset \neq [\alpha] \subseteq \|\beta\|$.

Closed confirmatory induction thus focuses more or less exclusively (apart from demanding α to be $[\cdot]$-consistent) on the logical weakness of β. Simple confirmatory consequence relations are syntactically characterised by Selectivity (called Right Consistency by Flach, 2000, p. 179), Right And (called And in Kraus, Lehmann, & Magidor, 1990, p. 179, and called Composition in Zwirn & Zwirn, 1996, p. 201), and Cut (called Predictive Right Weakening by Flach, 2000, p. 178).

xi.	$\alpha \mid\sim \beta,\ \alpha \mid\sim \gamma \ \Rightarrow\ \alpha \mid\sim \beta\wedge\gamma$	Right And
xii.	$\alpha \mid\sim \beta,\ \alpha\wedge\beta \mid\sim \gamma \ \Rightarrow\ \alpha \mid\sim \gamma$	Cut

As simple confirmatory consequence relations violate Left Logical Equivalence, Verification, and Right Reflexivity, they are no inductive consequence relations (though they do satisfy Right Logical Equivalence, Falsification, Left Reflexivity, Right Extension, and F-Consistency).

$W = \langle S, l, \prec\rangle$ is a preferential structure (cf. Kraus, Lehmann, & Magidor, 1990) iff l is a function from the set of states S into the set of all models $Mod_{\mathcal{L}}$, and $\prec$ is a strict partial order on S such that for all $\alpha \in \mathcal{L}$ and all $t \in \hat{\alpha} = \{s \in S: l(s) \vDash \alpha\}$: t is minimal w.r.t. $\prec$ or there is an $s \prec t$ which is minimal in $\hat{\alpha}$. A preferential structure $W = \langle S, l, \prec\rangle$ induces a *preferential confirmatory structure* by defining:

$$\|\alpha\| = \{s \in S: l(s) \vDash \alpha\}, \quad [\alpha] = \{s \in \|\alpha\|: \forall s' \in S: s' < s \rightarrow s' \notin \|\alpha\|\}.$$

Every preferential confirmatory structure is a simple confirmatory structure. Preferential confirmatory consequence relations, i.e. consequence relations $\mid\sim_W$

with W a preferential confirmatory structure, satisfy all principles for inductive consequence relations. They are syntactically characterised by Selectivity, Right And, Cut, and, in addition, Left Logical Equivalence, Confirmatory Reflexivity, Left Or (called Or in Kraus, Lehmann, & Magidor, 1990, p. 190), and Strong Verification.

32.	$\alpha \mathrel{\vert\!\sim} \alpha,\ \alpha \mathrel{\vert\!\not\sim} \neg\beta \ \Rightarrow\ \beta \mathrel{\vert\!\sim} \beta$	Confirmatory Reflexivity
xiii.	$\alpha \mathrel{\vert\!\sim} \gamma,\ \beta \mathrel{\vert\!\sim} \gamma \ \Rightarrow\ \alpha\vee\beta \mathrel{\vert\!\sim} \gamma$	Left Or
xiv.	$\alpha \mathrel{\vert\!\sim} \gamma,\ \alpha \mathrel{\vert\!\sim} \beta \ \Rightarrow\ \alpha\wedge\gamma \mathrel{\vert\!\sim} \beta$	Strong Verification

Assessment relations satisfy Selectivity, Left Logical Equivalence, and Confirmatory Reflexivity, but they violate Right And, Cut, Right Weakening, Admissible Entailment, Left Or, and Strong Verification.

In contrast to closed confirmatory consequence relations, *open confirmatory consequence relations* $\mathrel{\vert\!\sim}_W$, where W is a confirmatory structure, are given by: $\alpha \mathrel{\vert\!\sim}_W \beta$ iff $[\alpha]\cap\|\beta\| \neq \emptyset$. Classical confirmatory structures are simple confirmatory structures with $[\cdot] = \|\cdot\|$. So open classical confirmatory consequence is just classical consistency. It satisfies all principles for inductive consequence relations and is syntactically characterised by Predictive Convergence, Cut, F-Consistency, and Disjunctive Rationality, none of which are satisfied by assessment relations.

xv.	$\alpha\vee\beta \mathrel{\vert\!\sim} \gamma,\ \beta \mathrel{\vert\!\not\sim} \gamma \ \Rightarrow\ \alpha \mathrel{\vert\!\sim} \gamma$	Disjunctive Rationality

As open classical confirmatory induction satisfies both Predictive Convergence and Cut, it somehow combines aspects of explanatory induction on the one hand and confirmatory induction on the other hand. However, the resulting system is so weak that just about anything goes. After all, only logically incompatible statements do not confirm each other. In contrast to this, the combination of the plausibility and informativeness aspects achieved by assessment relations is much more stringent: In order for β to be a possible inductive consequence of α, β must be at least as plausible as and more informative than its negation $\neg\beta$, or more plausible than and at least as plausible as its negation $\neg\beta$ (relative in each case to α).

6.2 Assessment relations versus nonmonotonic consequence relations

The following principles are not satisfied by assessment relations (v–vii are from Zwirn & Zwirn, 1996; viii–xi are from Kraus, Lehmann, & Magidor, 1990).

xvi.	$\alpha \vdash \beta \ \Rightarrow\ \alpha \mathrel{\vert\!\sim} \beta$	Entailment, Supraclassicality*

xvii.	$\beta \vdash \alpha \;\Rightarrow\; \alpha \mid\!\sim \beta$	Conversion
xviii.	$\alpha \mid\!\sim \beta \;\Rightarrow\; \neg\beta \mid\!\sim \neg\alpha$	Contraposition
xix.	$\alpha \mid\!\sim \beta\rightarrow\gamma \;\Rightarrow\; \alpha\wedge\beta \mid\!\sim \gamma$	EHD
xx.	$\alpha \mid\!\sim \beta,\; \beta \mid\!\sim \gamma \;\Rightarrow\; \alpha \mid\!\sim \gamma$	Transitivity
xxi.	$\alpha \mid\!\sim \beta,\; \beta \mid\!\sim \alpha,\; \alpha \mid\!\sim \gamma \;\Rightarrow\; \beta \mid\!\sim \gamma$	Equivalence
xxii.	$\alpha\wedge\beta \mid\!\sim \gamma \;\Rightarrow\; \alpha \mid\!\sim \beta\rightarrow\gamma$	S

However, assessment relations do satisfy the following three KLM-principles.

33.	$\alpha \mid\!\sim \beta\rightarrow\gamma,\; \alpha \mid\!\sim \beta \;\Rightarrow\; \alpha \mid\!\sim \gamma$	MPC
34.	$\alpha_0 \mid\!\sim \alpha_1,\; \ldots,\; \alpha_{k-1} \mid\!\sim \alpha_k,\; \alpha_k \mid\!\sim \alpha_0 \;\Rightarrow\; \alpha_0 \mid\!\sim \alpha_k$	Loop
35.	$\alpha\wedge\beta \mid\!\sim \gamma,\; \alpha\wedge\neg\beta \mid\!\sim \gamma \;\Rightarrow\; \alpha \mid\!\sim \gamma$	Proof by Cases, D

The violation of the following principle (called Monotonicity in Kraus, Lehmann, & Magidor, 1990, p. 180) means that assessment relations are not monotonic.

xxiii.	$\beta \mid\!\sim \gamma,\; \alpha \vdash \beta \;\Rightarrow\; \alpha \mid\!\sim \gamma$	Left Monotonicity

However, assessment relations are *genuinely nonmonotonic* in the sense that not only Left, but also Right Monotonicity is not admissible.

xxiv.	$\alpha \mid\!\sim \beta,\; \beta \vdash \gamma \;\Rightarrow\; \alpha \mid\!\sim \gamma$	Right Monotonicity, Right Weakening

So not only arbitrary strengthening of the premises, but also arbitrary weakening of the conclusion is not allowed. The reason is this: By arbitrary weakening of the conclusion, information is lost – and the less informative conclusion might not be worth taking the risk of being led to a false conclusion.

Furthermore, the assessment approach can answer the question of why everyday reasoning is satisfied with a standard that is weaker than truth preservation in all possible worlds (e.g. truth preservation in all normal worlds), and thus runs the risk of being led to a false conclusion: We are willing to take this risk, because we want to arrive at informative conclusions that go beyond the premises. However, like relevance relations, assessment relations are no proper consequence relations in the sense that their semantics is not in terms of the preservation of a particular property.

7. Carnap's analysis revisited

In conclusion, let us turn back to Carnap's analysis of Hempel's conditions and his claim that Hempel was mixing up absolute and incremental confirmation. As argued in sections 2–4, Carnap's analysis is neither charitable nor illuminating, and there is a more charitable interpretation that is illuminating by accounting for Hempel's triviality result and his rejection of the Converse Consequence Condition. Still, one might be interested in the relation between Carnap's favoured concept of qualitative confirmation – viz. positive probabilistic relevance in the sense of a regular probability Pr – and our assessment relations leading to sufficiently plausible and informative conclusions.

As presented in section 5, assessment relations are unconditionally reflexive, whence any tautology is an acceptable theory for tautological data, and any contradiction is an acceptable theory for contradictory data. This is a consequence of stipulating $\kappa(A \mid B) = 0$ whenever $\kappa(B) = \infty$ and could have been avoided (as in Flach's approach). In contrast, positive (probabilistic or rank-theoretic) relevance on a field $\mathcal{A}$ over a non-empty set of possibilities W is reflexive except for tautological or contradictory propositions. The gap can be closed by extending the notion of positive relevance to include the pairs $\langle\emptyset, \emptyset\rangle$ and $\langle W, W\rangle$ so that tautologies are positively relevant for tautologies and contradictions are positively relevant for contradictions. Let us call this broadened notion *extended* positive relevance.

The relation between assessment and extended positive relevance is still slightly obscured by the fact that assessment relations so far have been characterised in terms of ranking functions, whereas Carnap's positive relevance account is probabilistic. Given the same framework, it is clear that extended positive relevance of α for β is a necessary condition for β to be an acceptable theory for α. More precisely, we have for any regular probability space $\langle W, \mathcal{A}, \Pr\rangle$, any regular ranking space $\langle W, \mathcal{A}, \kappa\rangle$, and any contingent $A, B \in \mathcal{A}$:

$$\left[\begin{array}{c}\Pr(B \mid A) > \Pr(\overline{B} \mid A)\\ \&\\ \Pr(\overline{B} \mid \overline{A}) \geq \Pr(B \mid \overline{A})\end{array}\right] \text{ or } \left[\begin{array}{c}\Pr(B \mid A) \geq \Pr(\overline{B} \mid A)\\ \&\\ \Pr(\overline{B} \mid \overline{A}) > \Pr(B \mid \overline{A})\end{array}\right] \Rightarrow \Pr(A \cap B) > \Pr(A) \cdot \Pr(B)$$

$$\left[\begin{array}{c}\kappa(B \mid A) < \kappa(\overline{B} \mid A)\\ \&\\ \kappa(\overline{B} \mid \overline{A}) \leq \kappa(B \mid \overline{A})\end{array}\right] \text{ or } \left[\begin{array}{c}\kappa(B \mid A) \leq \kappa(\overline{B} \mid A)\\ \&\\ \kappa(\overline{B} \mid \overline{A}) < \kappa(B \mid \overline{A})\end{array}\right] \Rightarrow \begin{array}{l}\kappa(A \cap B) + \kappa(\overline{A} \cap \overline{B}) <\\ < \kappa(A \cap \overline{B}) + \kappa(\overline{A} \cap B)\end{array}$$

where the last clause is the definition of positive κ-relevance of *A* for *B* (Spohn, 1999, p. 6). However, as

xxv. $\alpha \mathrel{|\!\!\sim} \beta \;\Rightarrow\; \beta \mathrel{|\!\!\sim} \alpha$ Symmetry

is not satisfied by assessment relations, the converse is not true. Both probabilistic and rank-theoretic (extended or unextended) positive relevance are symmetric, whereas assessment relations are not – which, as noted by Christensen (1999, p. 437f.), is as it should be.

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References

Carnap, R. (1962). *Logical Foundations of Probability,* 2nd ed. Chicago: University of Chicago Press.

Christensen, D. (1999). Measuring Confirmation. *Journal of Philosophy, 96,* 437–461.

Fitelson, B. (2001). *Studies in Bayesian Confirmation Theory.* PhD Dissertation. Madison: University of Wisconsin-Madison.

Flach, P.A. (2000). Logical Characterisations of Inductive Learning. In D.M. Gabbay & R. Kruse (Eds.). *Abductive Reasoning and Learning*. Dordrecht: Kluwer, 155–196.

Gabbay, D.M. (1985). Theoretical Foundations for Non-Monotonic Reasoning in Expert Systems. In K.R. Apt (Ed.). *Logics and Models of Concurrent Systems. NATO ASI Series, Vol. 13* (pp. 439–457). Berlin: Springer.

Goldszmidt, M., & Pearl, J. (1996). Qualitative probabilities for default reasoning, belief revision, and causal modeling. *Artificial Intelligence, 84,* 57–112.

Hempel, C.G. (1945). Studies in the Logic of Confirmation. *Mind, 54,* 1–26, 97–121. Reprinted in C.G. Hempel, 1965, *Aspects of Scientific Explanation and Other Essays in the Philosophy of Science* (pp. 3–51), New York: The Free Press.

Huber, F. (2004). *Assessing Theories. The Problem of a Quantitative Theory of Confirmation.* PhD Dissertation. Erfurt: University of Erfurt.

Huber, F. (in preparation). Assessing Theories, Bayes Style.

Kraus, S., Lehmann, D., & Magidor, M. (1990). Nonmonotonic Reasoning, Preferential Models, and Cumulative Logics, *Artificial Intelligence, 40,* 167–207.

Makinson, D. (1988). General Theory of Cumulative Inference. In M. Reinfrank, J. de Kleer, M.L. Ginsberg, & E. Sandewall (Eds.). *Non-Monotonic Reasoning*, *Lecture Notes in Artificial Intelligence, Vol. 346* (pp. 1–18). Berlin: Springer.

Milne, P. (2000). Is There a Logic of Confirmation Transfer? *Erkenntnis, 53,* 309–335.

Spohn, W. (1988). Ordinal Conditional Functions: A Dynamic Theory of Epistemic States. In W.L. Harper & B. Skyrms (Eds.). *Causation in Decision, Belief Change, and Statistics II* (pp. 105–134). Dordrecht: Kluwer.

Spohn, W. (1990). A General Non-Probabilistic Theory of Inductive Reasoning. In R.D. Shachter, T.S. Levitt, L.N. Kanal, & J.F. Lemmer (Eds.). *Uncertainty in Artificial Intelligence 4* (pp. 149–158). Amsterdam: North-Holland.

Spohn, W. (1999). Ranking Functions, AGM Style. In B. Hansson, S. Halldén, N.-E. Sahlin, & W. Rabinowicz (Eds.). *Internet Festschrift for Peter Gärdenfors*. Lund.

Zwirn, D., & Zwirn, H.P. (1996). Metaconfirmation. *Theory and Decision, 41,* 195–228.